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The exponential stability and stabilization of non-autonomous mechanical systems with non-conservative forces☆

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Abstract

The exponential stability of the unperturbed motion of a non-autonomous mechanical system with a complete set of forces, that is, dissipative, gyroscopic, potential and non-conservative positional forces, is investigated. The problem of stabilizing a non-autonomous system with specified non-conservative forces is considered with and without the use of potential forces. The problem of stabilizing a non-autonomous system with specified potential forces by the action of the forces of another structure is studied. The domain of stabilizability of the relative equilibrium position of a satellite in a circular orbit is found.
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1. Formulation of the problem

The equations of the perturbed motion of an extensive class of mechanical systems can be represented in the form[1–3]

$$A(t)\ddot{q} + (B(t) + G(t))\dot{q} + (C(t) + P(t))q = Q(t, q, \dot{q}) \tag{1.1}$$

where $q, \dot{q} \in R^n$ are the vectors of the generalized coordinates and velocities, the symmetric positive-definite matrix $A(t) = A^T(t) \gg 0$ contains the inertial characteristics of the system, the symmetric matrices $B(t) = B^T(t)$ and $C(t) = C^T(t)$ describe the dissipatively accelerating and the potential forces respectively, the skew-symmetric matrices $G(t) = -G^T(t)$ and $P(t) = -P^T(t)$ describe the gyroscopic and non-conservative positional forces respectively and the vector function $Q(t, q, \dot{q})$ denotes the set of non-linear terms, the expansions of which in power series of the generalized coordinates and velocities start with terms of not lower than the second degree. The trivial solution $q = \dot{q} = 0$ of system (1.1) corresponds to the unperturbed motion.

The problem of the effect of the structure of the forces on the stability of motion is one of the classical problems in the theory of stability and analytical mechanics.[1–3] This problem can be formulated as a problem of analysis, that is, of obtaining the conditions for the stability (asymptotic stability, instability, etc.) of the equilibrium state in terms of the invariant characteristics (trace, determinant, eigenvalues, etc.) of the matrix-coefficients occurring in the equations. If, however, only certain matrices are specified in the equations of motion and the remaining matrices can be chosen with the aim of guaranteeing a given dynamic property in a system, then a problem of synthesis arises and, in particular, the problem of stabilization, when it is required that the asymptotic stability of the equilibrium position be ensured.

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The case when the system is autonomous and there are no positional non-conservative forces, that is, when $P=0$, has been comparatively well studied. In this case, if the matrices of the dissipative and potential forces are positive-definite, $B\gg 0$, $C\gg 0$, then the equilibrium position is asymptotically stable (the third Thomson Tate Chetayev (TTCh) theorem). If, however, the matrix of the dissipative forces is positive-definite, $B\gg 0$, and the matrix of the potential forces C has just a single negative eigenvalue, then the equilibrium position is unstable (the fourth TTCh theorem). Note that the asymptotic stability is always exponential in the case of systems with constant coefficients. The third TTCh theorem therefore enables one to reveal the exponential stability of an equilibrium state, irrespective of the dependence on a specific form of the non-linear forces $Q(t, q, \dot{q})$.

In the case when there are non-conservative positional forces in the system, that is, when $P\neq 0$, the investigation of the stability becomes much more difficult and coordinates transformation have been proposed in a number of papers[4,5] which enable one to eliminate the non-conservative positional forces from the system. This opens up the possibility of using the TTCh theorems.[1–3] In certain cases, the use of this approach enables one to obtain effective conditions of asymptotic stability for classes of mechanical systems which are of important significance in practice.[6] However, it has correctly been noted[7] that the existence of this kind of transformations has only been proved with certain additional assumptions, that are not always satisfied. A second effective approach, which enables one to obtain stability conditions without eliminating non-conservative forces, involves the use of the method of Lyapunov functions and is successfully developed in Refs. 8–12. The first Lyapunov method, based on a study of the properties of the roots of the characteristic equation, is also used and continues to be developed in connection with mechanical systems with non-conservative forces.[9,10,13–17]

The main problem in this paper is to obtain the conditions for the exponential stability of the equilibrium position of system (1.1), which extend the third TTCh theorem to the case when there are non-conservative positional forces in the system. In addition, the problem of stabilizing the equilibrium position of a system of the form of (1.1) is considered, in which non-conservative positional forces play an important role, including a stabilizing role.

We shall use the following notation below

$$a = \inf\lambda_{\min}(A(t)),\quad a_{\max} = \sup\lambda_{\max}(A(t)),\quad a_* = \sup\lambda_{\max}(\dot{A}(t))$$

$$b = \inf\lambda_{\min}(B(t)),\quad b_{\max} = \sup\lambda_{\max}(B(t)),\quad b_* = \sup\lambda_{\max}(\dot{B}(t))$$

$$c = \inf\lambda_{\min}(C(t)),\quad c_{\max} = \sup\lambda_{\max}(C(t)),\quad c_* = \sup\lambda_{\max}(\dot{C}(t))$$

$$p_{\min} = \inf\sqrt{\lambda_{\min}(P^T(t)P(t))},\quad p = \sup\sqrt{\lambda_{\max}(P^T(t)P(t))}$$

$$p_* = \sup\sqrt{\lambda_{\max}(\dot{P}^T(t)\dot{P}(t))},\quad g = \sup\sqrt{\lambda_{\max}(G^T(t)G(t))}$$

Here and henceforth, the largest and the smallest eigenvalues of the matrix M are denoted by $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ and the operations sup and inf are carried out over all real values of $t\in R$.

2. The exponential stability of non-conservative systems

Theorem 2.1. *Suppose that, in system (1.1), the symmetric positive-definite matrices $A(t)$, $B(t)$, $C(t)$ of the inertial characteristics and the acting dissipative and potential forces respectively are such that the conditions*

$$a>0,\ b>0,\ c>0,\ a_{\max}<+\infty,\ b_{\max}<+\infty,\ c_{\max}<+\infty,\ a_*<+\infty,\ b_*<+\infty,\ c_*<+\infty$$

are satisfied and the skew-symmetric matrices $P(t)$ and $G(t)$ of the non-conservative positional forces and the gyroscopic forces are such that the conditions

$$p<+\infty,\quad g<+\infty$$

are satisfied.

The unperturbed motion

$$q = \dot{q} = 0 \tag{2.1}$$

will then be exponentially stable irrespective of the dependence of the non-linear forces $Q(t, q, \dot{q})$, if the inequalities

$$\begin{gathered} 2b > a_* \\ D = [2p(g+a_*) - 2a_{\max}c_* - (2b-a_*)(2c-b_*)]^2 - \\ -4[(2b-a_*)c_* + p^2][(g+a_*)^2 + 2a_{\max}(2c-b_*)] > 0 \\ \frac{(2b-a_*)(2c-b_*) + 2a_{\max}c_* - 2p(g+a_*) + \sqrt{D}}{2((2b-a_*)c_* + p^2)} > \frac{2a_{\max}}{2b-a_*} \end{gathered} \tag{2.2}$$

Proof. Consider the following quadratic form as the Lyapunov function

$$V(t, q, \dot{q}) = q^T(B + \mu C)q + q^T A\dot{q} + \dot{q}^T A q + \mu \dot{q}^T A \dot{q}$$

Here, $\mu > 0$ is a certain parameter, the value of which can be chosen and, for brevity, the explicit dependence of the matrices on time is not shown. The derivative of the functions $V(t, q, \dot{q})$, by virtue of the linear part of system (1.1), can be represented in the form

$$\begin{gathered} \dot{V}(t, q, \dot{q}) = -W(t, q, \dot{q}) = -q^T(2C - \dot{B} - \mu\dot{C})q - \\ -q^T(G - \mu T - \dot{A})\dot{q} - \dot{q}^T(G^T - \mu P^T - \dot{A})q - \dot{q}^T(2\mu B - 2A - \mu\dot{A})\dot{q} \end{gathered}$$

When account is taken of the theorem, we have the estimates

$$V(t, q, \dot{q}) \le (b_{\max} + \mu c_{\max})\|q\|^2 + 2a_{\max}\|q\|\|\dot{q}\| + \mu a_{\max}\|\dot{q}\|^2$$

$$V(t, q, \dot{q}) \ge (b + \mu c)\|q\|^2 - 2\|q\|\|A\dot{q}\| + \mu a_{\max}^{-1}\|A\dot{q}\|^2$$

$$W(t, q, \dot{q}) \ge (2c - b_* - \mu c_*)\|q\|^2 - 2(g + \mu p + a_*)\|q\|\|\dot{q}\| + (2\mu b - 2a_{\max} - \mu a_*)\|\dot{q}\|^2$$

Here and henceforth, the well-known inequalities

$$\lambda_{\min}(M)\|x\|^2 \le x^T M x \le \lambda_{\max}(M)\|x\|^2, \quad |x^T N y| \le \sqrt{\lambda_{\max}(N^T N)}\|x\|\|y\|$$

are used to estimate the quadratic forms and their derivatives according to the system. These inequalities hold for arbitrary vectors x and y, the matrix N and the symmetric matrix M. The norm of a vector is Euclidean everywhere.

Taking account of the fact that, under the conditions of the theorem, the positive-definiteness of $V(t, q, \dot{q})$ with respect to $||A\dot{q}||$ is equivalent to positive definiteness with respect to $||\dot{q}||$ and applying the Silvester criterion to quadratic forms of $||q||$ and $||A\dot{q}||$, $||q||$ and $||\dot{q}||$, we find the conditions imposed on the parameter μ which, when satisfied, ensure the positive definiteness of $V(t, q, \dot{q})$ and $W(t, q, \dot{q})$:

$$c\mu^2 + b\mu - a_{\max} > 0$$

$$(2b - a_*)\mu - 2a_{\max} > 0$$

$$(2c - b_* - c_*\mu)(2b\mu - 2a_{\max} - a_*\mu) - (g + \mu p + a_*)^2 > 0$$

This system of inequalities will have a positive solution $\mu > 0$, if conditions (2.2) are satisfied. Regardless of the non-linear forces, the exponential stability of the equilibrium position now follows from Krasovskii's theorem.[18]

We will now consider some special cases. □

Corollary 1. *In the case of constant positive-definite matrices A, B and C for the inertial characteristics, dissipative forces and potential forces respectively, the unperturbed motion (2.1) of system (1.1) will be exponentially stable irrespective of the dependence of the non-linear forces when the following inequality is satisfied*

$$p < \frac{2bc}{\sqrt{g^2 + 4ca_{\max}} + g} \tag{2.3}$$

Note that, when there are no conservative forces ($p = 0$), this inequality is automatically satisfied in the case of constant positive-definite matrices A, B and C in the case of any matrix of the gyroscopic forces.

The sufficient condition for the asymptotic stability of systems with constant coefficients, which is applicable to system (1.1) and is written using the notation adopted here in the form of (2.3), was obtained in Ref. 16 on the basis of a rigorous proof and analysis of the Metelitsyn inequality[13] (that is, within the framework of the first Lyapunov method). Hence, in the given case, both the first and second Lyapunov methods gave the same estimate (2.3). However, in the case of variable matrices $P(t)$ and $G(t)$ for the non-conservative and gyroscopic forces, use of the Metelitsyn method is now impossible since the negativeness of the real parts of the roots of the characteristic equation does not guarantee the stability of systems with variable coefficients, and Theorem 2.1 therefore extends the result presented above[16] to the case of non-autonomous systems.

Corollary 2. *If system (1.1) has the form*

$$h > h_0 = \frac{p}{2cb}(g + \sqrt{g^2 + 4c})$$

the constant matrices B and C are positive-definite and the conditions of Theorem 2.1 are satisfied for the matrices G(t) and P(t), then the unperturbed motion (2.1) will be exponentially stable, irrespective of the non-linear forces, for all sufficiently large values of the parameter

$$\ddot{q} + (B + G(t))\dot{q} + (hC + P(t))q = Q(t, q, \dot{q})$$

Note that the fact of asymptotic stability for all sufficiently large values of the parameter in the case of an autonomous system and a diagonal matrix C was established earlier in Ref. 9 using a different construction of the Lyapunov function. Corollary 2 extends and, under the corresponding assumptions[9] also supplements this result by producing an explicit estimate of the parameter from below.

Corollary 3. *If system (1.1) has the form*

$$\ddot{q} + (B + G(t))\dot{q} + (hC + $$

the constant matrices B and C are positive-definite and the conditions of Theorem 2.1 are satisfied in the case of the matrices G(t) and P(t), then the perturbed motion (2.1) will be exponentially stable, irrespective of the non-linear forces, for all sufficiently large values of the parameter

$$h > h_0 = \frac{p^2 + pbg}{cb^2}$$

We shall henceforth assume everywhere that system (1.1) is solved for the highest derivative, that is, we shall assume that $A(t) = \mathrm{E}$ is a unit matrix.

We will now consider system (1.1) under the assumptions that there are no non-linear and non-conservative forces, the matrices $G(t)$ and $B(t)$ are continuous and bounded, the matrix of the dissipative forces is positive-definite ($b > 0$) and the matrix of the potential forces is also positive-definite ($c > 0$). System (1.1) can then be written in the form

$$\ddot{q} + (B(t) + G(t))\dot{q} + Cq = 0 \tag{2.4}$$

Under the assumptions which have been made, it follows from Matrosov's theorem (Ref. 19, Theorem 4.4) that the unperturbed motion (2.1) of system (2.4) is uniformly asymptotically stable. This result cannot be obtained from

Theorem 2.1 since it does not contain any constraints which are imposed on the derivative However, Matrosov's theorem only ensures asymptotic and not exponential stability. It is therefore difficult, using this theorem, to estimate the rate of approach to equilibrium and the magnitude of the non-conservative forces which do not cause a breakdown of stability. In Matrosov's theorem the non-linear forces are not arbitrary but specified. These are potential, gyroscopic and resistance forces and the existence of non-conservative non-linear and/or linear forces in the system is not permitted.

Using a construction of the Lyapunov function of the form "total energy plus a small cross term", which has been successfully used earlier,[1,9,20] we will show that, under the conditions imposed by Matrosov's theorem on the linear dissipative, gyroscopic and potential forces, the equilibrium state (2.1) of the linear system (2.4) will be exponentially stable, and we will give an estimate of the linear non-conservative forces for which this property is preserved.

Theorem 2. *Suppose system (1.1) has the form*

$$\ddot{q} + (B(t) + G(t))\dot{q} + (C + \varepsilon P(t))q = Q(t, q, \dot{q}) \tag{2.5}$$

If the conditions

$$b > 0, \quad c > 0, \quad b_{\max} < +\infty, \quad g < +\infty, \quad p < +\infty$$

are satisfied, the equilibrium position (2.1) will be exponentially stable, regardless of the non-linear forces, for all values of the parameter $\varepsilon = \text{const}$ *which satisfy the inequality*

$$|\varepsilon| < \varepsilon_0 = \min\left\{\frac{\sqrt{c}}{2}, \frac{8bc}{16c + (2b_{\max} + 2g + p)^2}\right\} \tag{2.6}$$

Proof. Following the approach adopted earlier,[1,9,20] we will consider the quadratic form

$$V(q, \dot{q}) = q^T C q + 2\mu q^T \dot{q} + 2\mu \dot{q}^T q + \dot{q}^T \dot{q}$$

as the Lyapunov function. Here, $\mu = |\varepsilon| > 0$ is a positive parameter. By virtue of the linear part of system (2.5), the derivative of the function $V(q, \dot{q})$ can be represented in the form

$$\dot{V}(q, \dot{q}) = -W(t, q, \dot{q}) = -q^T(4\mu C)q - q^T(2\mu(B(t) + G(t)) + \varepsilon P^T(t))\dot{q} -$$
$$-\dot{q}^T(2\mu(B(t) + G(t))^T + \varepsilon P(t))q - 2\dot{q}^T(2B - 4\mu E)\dot{q}$$

When account is taken of the conditions of the theorem, we have the estimates

$$V(q, \dot{q}) \le c_{\max}\|q\|^2 + 4\mu\|q\|\|\dot{q}\| + \|\dot{q}\|^2, \quad V(q, \dot{q}) \ge c\|q\|^2 - 4\mu\|q\|\|\dot{q}\| + \|\dot{q}\|^2$$

$$W(t, q, \dot{q}) \ge 4\mu c\|q\|^2 - 2\mu(2b_{\max} - 2g + p)\|q\|\|\dot{q}\| + 2(b - 2\mu)\|\dot{q}\|^2$$

Applying Silvester's criterion to the quadratic forms of $||q||$ and $||\dot{q}||$, we find the conditions which the parameter μ must satisfy in order to ensure the positive definiteness of $V(q, \dot{q})$ and $W(t, q, \dot{q})$:

$$c - 4\mu^2 > 0, \quad 8\mu c(b - 2\mu) - \mu^2(2b_{\max} + 2g + p)^2 > 0$$

Both inequalities will be satisfied if the parameter ε satisfies condition (2.6). On applying Krasovskii's theorem[18] again, we obtain the exponential stability of the equilibrium position regardless of the non-linear forces. □

Example 1. We will now consider a mechanical system with two degrees of freedom, the equation of motion of which has the form

$$\ddot{q} + (E + pJ)\dot{q} + (E - pJ)q = 0; \quad J = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}, \quad q \in R^2$$

Here p is a real parameter and E, as before, is a unit matrix of the corresponding dimension. In the case of this example, Theorem 2.1 gives the sufficient condition of asymptotic stability in the form of the inequality $p^2 < 1/2$. Using the

Hurwitz criterion, it is easily verified that this condition is also a necessary condition. Note that Agafonov's theorem[11] is inapplicable in this example since the matrix

$$S = G^T P + P^T G + CG + G^T C = -2p^2 E$$

is not positive-definite.

In the case when $p = p(t)$ can change with time, the Hurwitz criterion cannot be used and, in accordance with Corollary 1, the sufficient conditions of exponential stability are given by the inequality $\sup p^2(t) < 1/2$. However, it is not known whether they are necessary conditions.

Example 2. The equations of motion of a rotor rotating in an aerodynamic medium, with steady cyclic damping which creates resistance forces with a common damping coefficient, have the form (Ref. 3, pp. 207–211)

$$\ddot{x} + b\dot{x} + k^2 x - py = X(x, y, \dot{x}, \dot{y}), \quad \ddot{y} + b\dot{y} + k^2 y + px = Y(x, y, \dot{x}, \dot{y})$$

The sufficient condition for asymptotic stability, which follows from Theorem 2.1, reduces in this example to the inequality $bk > p$ and is identical to the necessary and sufficient condition found in Ref. 3 on the basis of the Hurwitz criterion. Note that the quantity p is proportional to the square of the angular velocity of rotation of the rotor (Ref. 3, p. 209) and, therefore, if the rotor rotates non-uniformly (under conditions of gathering momentum), $p = p(t)$ will now not be a constant quantity and, consequently, the Hurwitz criterion cannot be used. For this case, Corollary 1 ensures exponential stability subject to the condition $bk > \sup p(t)$. In this example, it is also impossible to use Agafonov's theorem[11] since the matrix of the gyroscopic forces in the system is a null matrix.

3. Stabilization of systems with specified non-conservative forces

Suppose that, in system (1.1), the non-conservative positional forces are given but the other linear and non-linear forces are not. Then, the equations of motion have the form

$$\ddot{q} + P(t)q = 0 \tag{3.1}$$

and the equilibrium position (2.1) will be (Ref. 19, p. 223, Corollary 4.2) unstable for any skew-symmetric matrix $P(t)$. We will therefore now consider the question of the possibility of stabilizing the equilibrium position (2.1) of system (3.1) by a combination of potential, dissipative and gyroscopic forces to exponential stability, which is also preserved in the case of arbitrary non-linear forces. The problem consists of finding a method for choosing the matrices $B(t)$, $G(t)$, $C(t)$ such that the equilibrium position (2.1) of the system

$$\ddot{q} + (B(t) + G(t))\dot{q} + (C(t) + P(t))q = Q(t, q, \dot{q}) \tag{3.2}$$

is exponentially stable.

We will initially consider the case when, apart from strictly non-conservative forces, potential forces with a constant matrix $C \gg 0$ also act on the system or they can be chosen.

Theorem 3.1. *For any continuous bounded matrix of the non-conservative forces $P(t)$, the equilibrium position (2.1) of system (2.3) will be stabilized to exponential stability, regardless of the form of the non-linear forces $Q(t, q, \dot{q})$ if potential and dissipative forces with constant positive-definite matrices $C \gg 0$ and $B \gg 0$ (possibly, with as small as desired $b = \lambda_{\min}(B) > 0$) and $c = \lambda_{\min}(C) > 0$ and gyroscopic forces with a matrix $G(t) = \mu P(t)$ for all sufficiently large $\mu > \mu_0 = b^{-1} = \lambda_{\min}^{-1}(B) > 0$ are added to the system.*

The proof is carried out using the same Lyapunov function as in Theorem 2.1. A special choice of the gyroscopic forces enables one to refine the estimate of the derivative considerably by virtue of the system.

Note that the approach to stabilization explained earlier in Ref. 16, which is based on the use of the Metelitsyn inequality in a constructive form, is only applicable in the case of a constant matrix $P(t)$ and guarantees the possibility of stabilization only in the case of sufficiently large dissipative and potential forces (in the case of $b^2 c > 0$).

Example 3. Consider a mechanical system under the action of potential forces with common Poincaré stability coefficients, arbitrary non-conservative positional forces which are explicitly independent of time and arbitrary non-linear forces. The equations of motion have the form

$$\ddot{q} + (cE + P)q = Q(t, q, \dot{q}) \tag{3.3}$$

It is well-known that the equilibrium state (2.1) of system (3.3) will be unstable irrespective of the actual form of the non-linear forces $Q(t, q, \dot{q})$, which has been proved in Ref. 8 by constructing a function which satisfies Chetayev's theorem[1] on instability and it also follows from the existence of a root with a positive real part in the case of the characteristic equation.[3] We will consider the question of the stabilization by the use of dissipative and gyroscopic forces.

After the addition of gyroscopic and dissipative forces, the equations of motion take the form

$$\ddot{q} + (B + G)\dot{q} + (cE + P)q = Q(t, q, \dot{q}) \tag{3.4}$$

Note that. since system (3.3) is unstable when there are no velocity-dependent forces, stabilization cannot be accomplished using the method pointed out by Ivanov (Ref. 7, Proposition 1). It follows from Theorem 3.1 that the equilibrium state (2.1) will be stabilized to exponential stability if one takes the matrix of the dissipative forces as an arbitrary positive-definite matrix ($B \gg 0$) and the matrix of the gyroscopic forces as being proportional to the matrix of the non-conservative forces $G = \mu P$ with a sufficiently large coefficient of proportionality $\mu > \mu_0 = \lambda_{\min}^{-1}(B)$. If, in system (3.3), the matrix of the non-conservative forces $P = P(t)$ is a bounded function of time, such that $p < +\infty$, then the dissipative and gyroscopic forces, chosen in the manner indicated above, also ensure stabilization of the unperturbed motion up to exponential stability.

In the case of an odd number of coordinates, system (3.4) with the stabilizing control actions chosen as indicated above cannot be investigated for stability by Agafonov's method[11] since the matrix

$$S = G^T P + P^T G + CG + G^T C = 2\mu P^T P$$

will not automatically be positive-definite.

We will now consider stabilization without invoking potential forces.

Theorem 3.2. *In the case of an even number of coordinates $n = 2k$ when there are no potential forces $C(t) \equiv 0$, the unperturbed motion (2.1) of system (3.2) with a bounded matrix of the non-conservative positional forces $P(t)$ which satisfies the condition of positive definiteness*

$$q^T P^T(t) P(t) q \geq p_{\min}^2 \|q\|^2; \quad p_{\min} > 0$$

and has a bounded derivative $\dot{P}(t)$ can be stabilized up to exponential stability regardless of the form of the non-linear forces by choosing an arbitrary constant positive-definite matrix of the dissipative forces $B \gg 0$ and a special choice of the matrix of the gyroscopic forces

$$G(t) = \eta P(t); \quad \eta > \eta_0 = \lambda_{\min}^{-1}(B) > 0$$

Proof. As the Lyapunov function, we will consider the quadratic form

$$V(t, q, \dot{q}) = q^T B q + q^T (E + \gamma P^T)\dot{q} + \dot{q}^T (E + \gamma P) q + \eta \dot{q}^T \dot{q} \tag{3.5}$$

Here, $\gamma > 0$ and $\eta > 0$ are certain parameters, the values of which we shall choose later and, for brevity, the explicit dependence of the matrix P on time is not shown. By virtue of the linear part of system (3.2), the derivative of the function $V(t, q, \dot{q})$, when there are no potential forces, can be represented in the form

$$\dot{V}(t, q, \dot{q}) = -W(t, q, \dot{q}) = -2\gamma q^T P^T P q - q^T (G + \eta P^T + \gamma P^T (B + G) + \gamma \dot{P}^T)\dot{q} -$$
$$- \dot{q}^T (G^T + \eta P + \gamma (B + G)^T P + \gamma \dot{P}) q - 2\dot{q}^T (\eta B - E)\dot{q}$$

When account is taken of the conditions of the theorem, we have the estimates

$$V(t,q,\dot{q}) \le b_{\max}\|q\|^2 + 2(1+\gamma p)\|q\|\,\|\dot{q}\| + \eta\|\dot{q}\|^2$$
$$V(t,q,\dot{q}) \ge b\|\dot{q}\|^2 - 2(1+\gamma p)\|q\|\,\|\dot{q}\| + \eta\|\dot{q}\|^2 \quad (3.6)$$

$$W(t,q,\dot{q}) \ge 2\gamma p_{\min}\|q\|^T - 2\gamma(pb_{\max} + pg + p_*)\|q\|\,\|\dot{q}\| + 2(\eta b - 1)\|\dot{q}\|^2 \quad (3.7)$$

On applying Silvester's criterion to the quadratic forms $||q||$ and $||\dot{q}||$, we find the conditions which the parameters γ and η must satisfy in order to ensure the positive definiteness of $V(t, q, \dot{q})$ and $W(t, q, \dot{q})$:

$$b\eta - (1+\gamma p)^2 > 0, \quad 4\gamma p_{\min}(\eta b - 1) - \gamma^2(pb_{\max} + pg + p_*)^2 > 0$$

Both inequalities will be satisfied if the parameters are chosen as follows:

$$\eta > \eta_0 = \frac{1}{b}, \quad \gamma < \min\left\{\frac{\sqrt{b\eta} - 1}{p}, \frac{2\sqrt{p_{\min}(\eta b - 1)}}{pb_{\max} + pg + p_*}\right\}$$

Application of Krasovskii's theorem[18] on exponential stability completes the proof. □

Remarks.

1°. The unperturbed motion will also be exponentially stable in the case when the coefficient of proportionality $\eta(t)$ between the matrices of the gyroscopic and non-conservative forces is taken as a bounded, continuously differentiable function of time, satisfying the condition

$$2b\inf\eta(t) - \sup|\dot{\eta}(t)| > 2$$

In certain cases for specified non-conservative forces with a variable matrix $P(t)$, the choice of the variable coefficient $\eta(t)$ can enable one to use a constant matrix G of the gyroscopic forces, which simplifies their practical implementation.

2°. In the case of a system with constant coefficients, the condition $p_{\min} > 0$ reduces to non-degeneracy of the matrix of the non-conservative positional forces $\det P \neq 0$. If, however, the matrix $P = \text{const}$ is degenerate, it is impossible to ensure the asymptotic stability of the linear system by choosing the dissipative and gyroscopic forces when there are no potential forces, since the equilibrium position (2.1) will not be isolated. Merkin[2,3,21] has pointed out the importance of the condition of the non-degeneracy of the matrix of the non-conservative forces, without which it is impossible to satisfy the condition of the Hurwitz criterion. We emphasize that both in Metelitsyn's theorem (Ref. 13, Theorem 3) as well as in Merkin's theorem (Ref. 3, p.201, Theorem 8), one is talking essentially only about *necessary* conditions for stabilization. For stabilization, it is *necessary* simultaneously to add both gyroscopic and dissipative forces but, as was correctly pointed out in Ref. 22, this does not imply at all that asymptotic stability will be achieved by the application of any forces of such a structure.

Theorem 3.2 not only confirms that stabilizing forces of the above-mentioned structure (non-zero gyroscopic and non-zero dissipative forces) are necessarily found[22] but it also provides a constructive method for forming the corresponding matrices, which is also applicable in the case of a variable matrix of the non-conservative positional forces $P(t)$. Other methods can also be used to choose the stabilizing matrices but they must all be based on the sufficient conditions for asymptotic stability.

3°. Theorem 3.2 extends and refines Agafonov's theorem (Ref. 8, Theorem 5) which was formulated for linear systems with constant coefficients in the case of a diagonal matrix $B = \text{diag}\{b_i\}$ and proved on the basis of Matrosov's theorem[23] using a Lyapunov function, which has just a constant-sign derivative. The condition $\det P \neq 0$ does not participate explicitly in the formulation of Agafonov's theorem but it is required in order to prove the above-mentioned method[8] and it was certainly assumed that it is satisfied. Note that, although Agafonov's theorem[8] was only formulated for a constant matrix $P(t) = \alpha G = \text{const}$, the proof presented in Ref. 8

also enables one to use it when this matrix is variable and satisfies the inequalities $p_{\min} > 0$, $p < +\infty$ (no constraints whatsoever are imposed on the derivative $\dot{P}(t)$). However, in this case, it is only possible to confirm stability but not the exponential stability of the linearized system, which does not allow one to transfer the result to the case of arbitrary non-linear forces, unlike Theorem 3.2. Hence, the approaches used in Theorem 3.2 and in Agafonov's theorem[8], are mutually complementary.

Example 4. A vertical gyroscope with a radial correction is described by the equations

$$J\ddot{\alpha} + b\dot{\alpha} - H\dot{\beta} - \chi\beta = X_1, \quad J\ddot{\beta} + b\dot{\beta} + H\dot{\alpha} + \chi\alpha = X_2$$

where all of the notation corresponds to that adopted earlier (Ref. 3, p.212) and with the description of the physical meaning of all of the parameters and the phase coordinates. Applying Theorem 3.2 to this system, we obtain the sufficient condition for asymptotic stability

$$\chi < bH/J \tag{3.8}$$

which is identical to the necessary and sufficient condition obtained using the Hurwitz criterion (Ref. 3, p. 213).

In the equations of the vertical gyroscope, the quantity χ is assumed to be constant and is the coefficient of proportionality between the measured angle of rotation of one end of the gimbals and the moment, applied to the other end, which is created by the corresponding slave motor. We assume that the gain slope of the characteristic of the slave motors, which create the forces of radial correction, can be varied during the course of the control process, for example, on account of oscillations in the power of the constant current supply circuit. This leads to the need to put $\chi = \chi(t) \neq \text{const}$ in the equations. Applying Theorem 3.2 for this case and taking account of Remark 1°, we arrive at the following conditions of exponential stability

$$0 < \chi_{\min} \le \chi(t) \le \chi_{\max} < \frac{bH}{J}, \quad |\dot{\chi}(t)| \le \dot{\chi}_{\max} \le 2J\left(\frac{bH}{J\chi_{\max}} - 1\right)\frac{\chi_{\min}^2}{H}$$

In the case when $\chi(t) = \text{const}$, these conditions become condition (3.8).

4. Stabilization of systems with specified potential forces

We will now consider the problem of stabilization up to exponential stability of the equilibrium position

$$q = 0, \quad dq/d\tau = 0 \tag{4.1}$$

of the potential system

$$d^2q/d\tau^2 + (C(\tau)q = 0 \tag{4.2}$$

with a continuous bounded variable matrix $C(\tau)$, which satisfies the condition

$$\|C(\tau)\| \le c_{\max} < +\infty \tag{4.3}$$

by adding of dissipative, non-conservative and gyroscopic forces. We will express the forces which are added for stabilization in terms of a large positive parameter $h > 0$ such that the equations of the perturbed motion are written in the form of the system

$$\frac{d^2q}{d\tau^2} + h(B+G)\frac{dq}{d\tau} + (C(\tau) + h^2P)q = 0 \tag{4.4}$$

In this system, we make the substitution $t = h\tau$, and the stabilization problem then reduces to choosing the constant matrices B, G and P of the dissipative, gyroscopic and non-conservative forces respectively such that the equilibrium position $q = \dot{q} = 0$ of the system

$$\ddot{q} + (B+G)\dot{q} + (\varepsilon^2C(\varepsilon t) + P)q = 0 \tag{4.5}$$

with a small parameter $\varepsilon^2 = h^{-2} > 0$ is exponentially stable in the case of a variable matrix of the potential forces.

Theorem 4.1. *In the case of an even number of coordinates, the equilibrium position $q = \dot{q} = 0$ of system (4.5), with a bounded matrix of the potential forces and a sufficiently small value of the parameter $\varepsilon^2 > 0$, can be stabilized up to exponential stability by choosing the constant, positive-definite matrix of the dissipative forces $B \geqslant 0$, the constant non-degenerate matrix of the non-conservative forces P and the constant matrix of the gyroscopic forces $G = \eta P$, which is proportional to it, with a sufficiently large value of the coefficient of proportionality $\eta > \eta_{\min} = 1/b > 0$.*

Proof. As the Lyapunov function, we will consider the quadratic form $V(q, \dot{q})$, which differs from expression (3.5) by the replacement of the parameter γ by ε. We shall assume that the matrix $B \geqslant 0$ is an arbitrary, positive-definite matrix, P is non-degenerate and the value of the parameter $\eta > 0$ can be chosen. By virtue of system (4.5), the derivative of the function $V(q, \dot{q})$ can be represented in the form

$$\dot{V}(t, q, \dot{q}) = -W(t, q, \dot{q}) = -q^T(2\varepsilon P^T P + 2\varepsilon^2 C(\varepsilon t) + \varepsilon^3 C(\varepsilon t)P - \varepsilon^3 PC(\varepsilon t))q -$$

$$-q^T(G + \eta P^T + \eta\varepsilon^2 C(\varepsilon t) + \varepsilon P^T B + \varepsilon P^T G)\dot{q} -$$

$$-\dot{q}^T(G^T + \eta P + \eta\varepsilon^2 C(\varepsilon t) + \varepsilon BP + \varepsilon G^T P)q - 2\dot{q}^T(\eta B - E)\dot{q}$$

We choose the matrix of the gyroscopic forces as follows: $G = \eta P$. Then, for the function $V(q, \dot{q})$, we have the estimates, which differ from (3.6) by the replacement of γ by ε, and, for the function $W(t, q, \dot{q})$, we obtain the estimate

$$W(t, q, \dot{q}) \geq (2\varepsilon p_{\min}^2 - 2\varepsilon^2 c_{\max} - 2\varepsilon^3 p_{\max} c_{\max})\|q\|^2 -$$

$$-2(\varepsilon p_{\max} b_{\max} + \eta\varepsilon^2 c_{\max} + \varepsilon\eta p_{\max}^2)\|q\|\|\dot{q}\| + 2(\eta b - 1)\|\dot{q}\|^2$$

Applying Silvester's criterion to the quadratic forms of $||q||$ and $||\dot{q}||$, we find the conditions which the parameters ε and η must satisfy in order to ensure the positive definiteness of $V(q, \dot{q})$ and $W(t, q, \dot{q})$:

$$b\eta - (1 + \varepsilon p_{\max})^2 > 0$$

$$4\varepsilon(b\eta - 1)(p_{\max}^2 - \varepsilon c_{\max} - \varepsilon^2 p_{\max} c_{\max}) - \varepsilon^2(p_{\max} b_{\max} + \varepsilon\eta c_{\max} + \eta p_{\max}^2)^2 > 0$$

Both inequalities will be satisfied when $\eta > \eta_{\min} = 1/b$ if the parameter $\varepsilon > 0$ is chosen to be sufficiently small. For example, it is possible to put

$$\eta = \frac{2}{b}, \quad \varepsilon = \min\left\{1, \frac{\sqrt{2} - 1}{p_{\max}}, \frac{4mp_{\max}^2}{4bc_{\max}(1 + b_{\max}) + bp_{\max}b_{\max} + 2c_{\max} + 2p_{\max}^2}\right\}$$

Application of Krasovskii's theorem[18] on exponential stability completes the proof. □

Remark. The stabilizing dissipative, gyroscopic and non-conservative positional forces, constructed in the course of the proof of Theorem 4.1 need not be included in the information concerning specific elements of the matrix $C(t)$ and, at once, stabilize the whole class of systems which satisfy condition (4.3).

Since the parameter $h > 0$ in system (4.4) can be chosen, the following assertion follows from Theorem 4.1.

Theorem 4.2. *In the case of an even number of coordinates, the equilibrium position (4.1) of system (4.2) with a bounded matrix of the potential forces can be stabilized everywhere up to exponential stability by the addition of dissipative, gyroscopic and non-conservative forces.*

Stabilization is impossible in the case of system (4.2) with a constant negative-definite matrix $C \ll 0$ and an odd number of degrees of freedom according to part 1 of Merkin's theorem (Ref. 3, p. 202, Theorem 9). According to part 2 of the same theorem, the simultaneous addition of gyroscopic and non-conservative positional forces is *necessary* for stabilization in the case of an even number of coordinates and a positive-definite matrix of the dissipative forces $B \gg 0$. However, neither the basic possibility of stabilization, the sufficient conditions for stabilizability, nor the method of choosing the stabilizing matrices for this case are discussed in the above-mentioned Merkin's theorem and they do not follow from it. These questions have been considered for systems with constant coefficients both in the case of the predominance of gyroscopic forces in the system[2] and in the general case.[7,14,15]

We will now consider the problem of stabilizing the equilibrium position (4.1) of the linear system (4.2) with a variable matrix $C(t)$ of the potential forces and an odd number of coordinates up to exponential stability by the addition of forces with a different structure.

Theorem 4.3. *In the case of an odd number of coordinates, the equilibrium position* (4.1) *of system* (4.2) *can be stabilized up to exponential stability by adding dissipative, gyroscopic and non-conservative forces in the case when just a single diagonal element of the matrix* $C(t)$ *differs from zero by a positive number.*

Proof. Without loss in generality, it can be assumed that

$$c_{11}(t) \geq c_{11}^* > 0$$

since this can always be achieved by renumbering the coordinates. We put

$$b_{1j}(t) = g_{1j}(t) = 0, \quad p_{1j}(t) = -c_{1j}(t); \quad j = 2, 3, \ldots, n$$

With such a choice of the gyroscopic, dissipative and non-conservative positional forces, the equation for the first coordinate is separated from the remaining equations and takes the form

$$\ddot{q}_1 + b_{11}\dot{q}_1 + c_{11}(t)q_1 = 0; \quad c_{11}(t) \geq c_{11}^* > 0 \tag{4.6}$$

By choosing the constant damping factor $b_{11} > 0$ to be sufficiently large, it is possible, as follows from Merkin's results (Ref. 3, p. 230), to attain exponential stability of the solution $q_1 = \dot{q}_1 = 0$ of Eq. (4.6).

For the vector of the remaining coordinates $y = \mathrm{col}(q_2, \ldots, q_n)$, the equations of the perturbed motion can be represented in the form

$$\ddot{y} + (B_1 + G_1)\dot{y} + (C_1(t) + P_1)y + u(t)q_1 = 0 \tag{4.7}$$

Here the $(n-1) \times (n-1)$ matrices B_1, G_1 and P_1 have the structure of the dissipative, gyroscopic and non-conservative forces respectively and can be chosen, the symmetric $(n-1) \times (n-1)$ matrix $C_1(t)$ is composed of the corresponding elements of the initial matrix $C(t)$ of the potential forces and the $(n-1)$-dimensional vector $u(t)$ depends on the elements of the first column of the initial matrix $C(t)$ of the potential forces. In accordance with Theorem 4.2, the isolated subsystem

$$\ddot{y} + (B_1 + G_1)\dot{y} + (C_1(t) + P_1)y = 0 \tag{4.8}$$

with an even number of coordinates can be stabilized up to exponential stability by the choice of the matrices B_1, G_1 and P_1.

The exponential stability of the coupled block-diagonal system (4.6), (4.7) now follows from the fact of the exponential stability of the isolated subsystems (4.6) and (4.8). □

When account is taken of the law of inertia of quadratic forms, Theorem 4.3 can be reformulated in the following manner in the case of a constant matrix C of the potential forces.

Theorem 4.4. *In the case of an odd number of coordinates and a constant matrix of the potential forces* C*, the equilibrium position* (4.1) *of the linear system* (4.2) *can be stabilized up to exponential stability by the addition of dissipative, gyroscopic and non-conservative forces with constant coefficient matrices if and only if the matrix* C *has just a single positive eigenvalue. Stabilization of this kind is always feasible in the case of an even number of coordinates and a constant matrix of the potential forces* C.

Proof. *Sufficiency*. The second assertion follows from Theorem 4.2. In order to prove the first assertion, we reduce the system to normal coordinates[1,2] using an orthogonal transformation and, if required, we renumber the coordinates. It is therefore possible, without loss of generality, to assume that the matrix C is diagonal and its diagonal elements are arranged in decreasing order. It then follows from the conditions of the theorem that $C_{11} > 0$, and the result follows from the preceding theorem.

The *necessity* follows from the fact that, in the case of a diagonal matrix C of odd order with non-positive elements on the principal diagonal and in the case of any skew-symmetric matrix P, the determinant $\det(C + P)$ will be non-positive, which can be shown by a literal repetition of Merkin's arguments when proving the properties of the determinants of skew-symmetric matrices (Ref. 2, pp.136–138) or by taking the limit in them with respect to the diagonal elements which vanish. □

Remark. The results formulated in Theorem 4.4 is not essentially new. It was established earlier in Ref. 15 in similar terms on the basis of the Hurwitz criterion and the results obtained by Lakhadanov.[12,14] However, the construction of the stabilizing control, proposed above in Theorem 4.1, differs fundamentally from that used earlier.[15] It does not require the use of information concerning specific elements of the matrix C and is applicable to the case of a variable matrix of the potential forces in system (4.2).

5. Stabilization of the relative equilibrium position of a satellite

The equation's of motion which have been linearized in the neighbourhood of the relative equilibrium $q = \dot{q} = 0$ (Ref. 24, p.54) have the form

$$\ddot{q} + M\dot{q} + Cq = 0$$

Here, $q^T = (\theta_1, \theta_2, \theta_3)$ is the vector of the generalized coordinates in the initial Beletskii notation, and the matrices C and M of the potential forces and the velocity-dependent forces are as follows:

$$C = \operatorname{diag}\{c_{11}, c_{22}, c_{33}\}, \quad M = \left\| \begin{matrix} 0 & c_{11} - 1 & 0 \\ 1 - c_{22}/4 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|$$

$$c_{11} = \frac{\bar{B} - \bar{A}}{\bar{C}}, \quad c_{22} = 4\frac{\bar{B} - \bar{C}}{\bar{A}}, \quad c_{33} = 3\frac{\bar{A} - \bar{C}}{\bar{B}}$$

The elements of the matrices are expressed in terms of the principal central moments of inertia of the satellite for which the initial notation[24] has had dashes on top in order to distinguish them from the notation used for the matrices in system (1.1).

We will assume that the effect of the forces $-M\dot{q}$, which solely depend on the velocities, is completely annulled by means of the addition of compensating forces $+M\dot{q}$ and, following Ivanov,[7] we will consider the problem of stabilizing the relative equilibrium position of a satellite in a circular orbit up to asymptotic stability by the addition of dissipative, gyroscopic and strictly non-conservative forces. The methods used in the practical realization of the above-mentioned forces are not touched upon here. A brief discussion of them can be foung in Ref. 7.

It was shown by Beletskii[24] that, when the inequalities

$$\bar{B} > \bar{A} > \bar{C} \tag{5.1}$$

are satisfied for the moments of inertia, the equilibrium state is stable (not asymptotically) with respect to the complete non-linear equations of motion. For the present, we shall assume that all the moments of inertia are different. In the case when, instead of conditions (5.1), the inequalities

$$\bar{B} < \bar{A} < \bar{C} \tag{5.2}$$

hold, stabilization is impossible, which follows from Part 1 of Melkin's theorem (Ref. 3, p. 202, Theorem 9) since, in this case, the matrix of the potential forces C is negative-definite and the number of coordinates is odd.

We will show that, in cases when at least one of inequalities (5.2) is coarsely violated, the problem of stabilization is solvable.

Actually, if all of the moments of inertia are different and conditions (5.2) are not satisfied, then at least one of inequalities (5.1) will necessarily be satisfied, and this means that one of the diagonal elements of the matrix C will be positive. We now add dissipative forces with a diagonal, positive-definite matrix B to the system, and, then, that component, to which the above mentioned positive diagonal element of matrix C corresponds, becomes asymptotically stable with respect to "its own" linearized equation which is isolated, by virtue of the diagonality of the matrices B and C, from the equations with (possibly) unstable components. By now adding dissipative, gyroscopic and non-conservative forces, which solely depend on the corresponding coordinates and velocities, to the remaining two equations, it is possible, on the basis of Theorem 4.2, to ensure asymptotic stability with respect to the two remaining coordinates and velocities.

Suppose, for example, that the moments of inertia are such that $\bar{A} = 10\Delta$, $\bar{B} = 11\Delta$, $\bar{C} = 12\Delta$, where $\Delta > 0$ is the difference in the magnitudes of the two closest moments of inertia. Then, the trace of the matrix C is equal to $569/660 < 0$ and stabilization on the basis of Ivanov's propositions (Ref. 7, Propositions 1 and 2) is not feasible. Applying the approach presented above, the stabilizing matrices can be chosen for the case being considered in the form

$$B = \text{diag}(b_1, b_2, b_3), \quad P = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & -p \\ 0 & p & 0 \end{Vmatrix}, \quad G = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & -g \\ 0 & g & 0 \end{Vmatrix}$$

where $b_i > 0$, p and g are certain parameters.

We will now consider the case when the satellite possesses dynamic symmetry. Simple inspection reveals that there is a whole family of such different cases: the case of a spherical satellite and a further six cases when two of any of the moments of inertia are identical and the third moment is greater or less than them. It can be shown that, in four cases, there will necessarily be just a single positive element among the diagonal elements of the matrix C, and, hence, stabilization by means of the method described above, based on Theorem 4.4, is possible in all these cases. The remaining three cases are obtained from conditions (5.2) by replacing one or both of the signs of strict inequality by the sign of equality. In all these cases, all the diagonal elements of the matrix C will be non-positive and, moreover, at least one of them is necessarily zero. Consequently, stabilization cannot be accomplished either on the basis of Theorem 4.4 or on the basis of Ivanov's method[7] and this is not fortuitous.

Actually, for $n = 3$ in the case of an arbitrary diagonal matrix C with the above mentioned properties and an arbitrary skew-symmetric matrix P, we obtain

$$\det(C + P) = c_{11}p_{23}^2 + c_{22}p_{13}^2 + c_{33}p_{12}^2 \le 0$$

It therefore follows from Melkin's theorem (Ref. 3, p. 201, Theorem 7) that the stabilization of the relative equilibrium position of a satellite in such cases, if it is possible, is only feasible in cases of a single zero root, which are critical in Lyapunov's sense and requires the use of appropriate methods of investigation,[25,26] taking account of the non-linear forces occurring in the complete equations of motion.

The compensating forces $+M\dot{q}$, with respect to their structure, are the sum of dissipative-accelerating forces and gyroscopic forces. Since the element of the diagonal matrix $B = \text{diag}\{b_i\}$ can be chosen, in accordance with Theorem 4.4, to be fairly large and positive, by taking them so that the matrix $B - (M + M^T)/2$ is positive-definite, we obtain that stabilization, when account is taken of compensation, is only achieved by non-conservative, dissipative and gyroscopic forces.

Hence, Theorem 4.4 enables one to carry out a complete analysis of the possibility of stabilizing the relative equilibrium position of a satellite in the linear approximation by using dissipative, gyroscopic and non-conservative positional forces: in the case of conditions (5.2), stabilization is impossible. In cases when just one of the inequalities becomes an equality, the possibility of stabilization must be investigated taking account of non-linear forces and, in all of the remaining cases, stabilization is possible.

The domains of stabilizability of the relative equilibrium position of a satellite in the space of the parameters, that is, in the plane of the ratios of the principal central moments of inertia, are shown in Fig. 1. Conditions (5.1) are satisfied in domain I and, for stabilization, it is sufficient to add some dissipative forces. In domain II, conditions (5.2) are satisfied and stabilization is impossible. On the broken line (0.0)–(1.1)–(1.0), which specifies the boundary of domain II, stabilization, if it is so, is only so in cases which are critical in Lyapunov's sense. In all of the remaining domains, Theorem 4.4 ensures that stabilization is possible by means of non-conservative, gyroscopic and dissipative forces. Stabilization by Ivanov's method[7] is possible above and to the right of the curve (0.1)–(1.1)–(1.0). The half-strip, where the parameters of the system can be interpreted as ratios of the moments of inertia, is separated out by the dot-dash line which passes through unity on the coordinate axes.

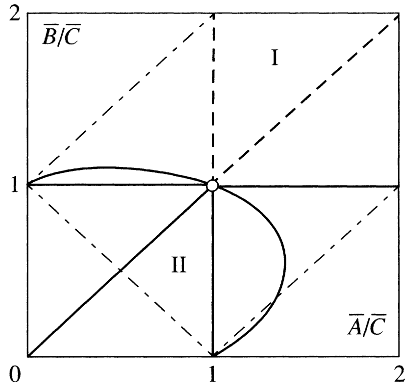

Fig. 1.

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